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ON THE GENERALIZED FREQUENCY FUNCTIONS OF EDGEWORTH.

I. INTRODUCTION.

While many distributions follow the normal law, if allowance is made for deviations due to random sampling, there are well known classes of variates which do not follow this law. The distribution of statures of certain classes of men fits the normal curve. It is only reasonable to expect that if linear measurements follow the normal law, the corresponding similar surfaces and volumes should be distributed in accord with some transformation of that law.

Let x_1, x_2, x_3, ..., x_n be variates of a distribution where

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}}$$

If X_1, X_2, X_3, ..., X_n are a new system of variates where

$$X_1 = kx_1, \quad X_2 = kx_2, \quad \ldots\ldots, \quad X_n = kx_n,$$

the type of the curve is not changed, as this transformation alters only the scale or the size of the modulus.

If, however, a function, other than a linear function of the variates is substituted for each variate, the form of the curve will be changed. For example, if we make the transformations

$$X_1 = kx_1^2, \quad X_2 = kx_2^2, \quad X_3 = kx_3^2, \quad \ldots\ldots, \quad X_n = kx_n^2,$$

and $$X_1 = kx_1^3, \quad X_2 = kx_2^3, \quad X_3 = kx_3^3, \quad \ldots\ldots, \quad X_n = kx_n^3,$$

the transformed frequency distributions may be regarded as distributions of surfaces and volumes. It is but natural to assume that certain distributions are in the nature of similar surfaces and volumes whose linear dimensions follow the normal law. In more general terms, it is but natural

to expect that certain observed variates are functions of more fundamental elements where those fundamental elements are normally distributed.

This conception of transformation of variates is fundamental in Edgeworth's "Method of Translation", and in certain theories which he has advanced concerning generalized frequency curves*.

It is a purpose of this paper to test the practicability of regarding some observed distributions as transformations of more elementary variates which follow the normal curve and to investigate the characters of some functions that are obtained by simple transformations of a normal distribution.

More precisely, if x_1, x_2, ..., x_n are a set of elementary variates that are normally distributed, new variates X_1, X_2, ..., X_n may be formed which are functions of these more elementary variates, say of the form

$$X_r = a_1 x_r + a_2 x_r^2 + \ldots\ldots + a_n x_r^n + \ldots\ldots \qquad (1).$$

The X's would of necessity follow some law of frequency. It may be of interest to determine the law for certain cases. The cases in which only a few terms of series (1) need be used are perhaps of the most interest. If a_2, a_3, ..., a_n, ... are small as compared with a and we let $\kappa = \frac{a_2}{a_1}$; $\lambda = \frac{a_3}{a_1}$; ...., we may write $X_r = a_1(x_r + \kappa x_r^2 + \lambda x_r^3 + \ldots\ldots)$.

It is not usually necessary to go beyond the third power in x provided κ and λ are small decreasing numbers as they are in the numerical illustration of section V. Or conversely, having an observed distribution given, we shall inquire into the consequences of treating it as a distribution which is formed by transformation from a certain fundamental normal distribution.

* Fifth International Congress of Mathematicians, vol. II, page 426, 1912. Journal of the Royal Statistical Society, 1898-1900.

II. TRANSFORMATION $X = x'^2$.

This is a very special type of transformation which surfaces might be expected to follow, since similar surfaces are to each other as the squares of their linear dimensions.

Let $f(X)dX$ be this distribution formed by squaring each variable of a system that follows the normal curve of standard deviation σ and center of gravity at $x=a$. Then the integral

$$\int f(X)dX = \frac{1}{\sigma\sqrt{2\pi}}\int e^{-\frac{(x-a)'^2}{2\sigma^2}}dx$$

gives the probability that a variable x of the generating system falls between assigned limits of integration. Since $X=x^2$, $dX=2xdx$, and $dx=\frac{dX}{2\sqrt{X}}$. Substituting $\sqrt{X}$ for x and $\frac{dX}{2\sqrt{X}}$ for dx in the above equation,

$$\int f(X)dX = \frac{1}{\sigma\sqrt{2\pi}}\int e^{-\frac{(\sqrt{X}-a)^2}{2\sigma^2}}\cdot\frac{1}{2\sqrt{X}}dX.$$

Since each element of the original group is squared to form the transformed distribution,each negative element of the normal distribution will become positive when squared. For this reason,the area under the normal curve is taken between the limits,say from $-\alpha$ to $+\alpha$, where α is any positive number, and the **area** under the transformed curve is taken from 0 to α^2. By hypothesis,α is any positive number, and we may extend the limits to infinity for convenience of integration. Then

$$\frac{1}{\sigma\sqrt{2\pi}}\int_0^{\infty} e^{-\frac{(\sqrt{X}-a)^2}{2\sigma^2}}\cdot\frac{1}{2\sqrt{X}}dX = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-a)'^2}{2\sigma^2}}dx.$$

As the unknown origin O is at a distance a from the center of gravity of the normal curve, a'^2 is the distance from the median of the transformed curve to the same origin O, provided a is large enough so that none of the variates of the normal curve are below the origin.

Let g be the distance from the origin to the mean of the observed curve and let $\sigma\sqrt{2}$ be the modulus of the normal curve.

By method of moments we shall determine the three constants. Since each variate x of the normal curve is replaced by x^2, we have

$$\int_0^\infty X^n f(X)dX = \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty X^n e^{-\frac{(\sqrt{X}-a)^2}{2\sigma^2}} \cdot \frac{1}{2\sqrt{X}}dX$$
$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} x^{2n} e^{-\frac{(x-a)^2}{2\sigma^2}} dx,$$

from which we see that we can use our knowledge of the normal curve in valuing the moments.

First moment. $$g = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} x^2 e^{-\frac{(x-a)^2}{2\sigma^2}} dx$$
$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} (x'+a)^2 e^{-\frac{x'^2}{2\sigma^2}} dx'$$

when we put $x = x' + a$. Integrating by parts

$$g = a^2 + \sigma^2 \;\ldots\ldots \qquad (1).$$

The mean, therefore, is equal to the median plus σ^2.

Second moment. The second moment coefficient about the center of gravity being denoted by μ_2, that about the origin is,

$$g^2 + \mu_2 = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} (x+a)^4 e^{-\frac{x^2}{2\sigma^2}} dx.$$

Integrating by parts gives $g^2 + \mu_2 = a^4 + 6a^2\sigma^2 + 3\sigma^4 \ldots\ldots \qquad (2)$

Third moment. The third moment coefficient about the center of gravity is μ_3, that about the origin is,

$$g^3 + 3g\mu_2 + \mu_3 = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} (x+a)^6 e^{-\frac{x^2}{2\sigma^2}} dx,$$
$$= a^6 + 15a^4\sigma^2 + 45a^2\sigma^4 + 15\sigma^6 \ldots\ldots \qquad (3)$$

By substituting the value of g from equation (1) into (2) and (3), we get

$$\mu_2 = 4a^2\sigma^2 + 2\sigma^4 \;\ldots\ldots \qquad (4)$$

$$\mu_3 = 24a^2\sigma^4 + 8\sigma^6 \;\ldots\ldots \qquad (5)$$

Eliminating a from (4) and (5)

$$4\sigma^6 - 6\sigma^2\mu_2 + \mu_3 = 0.$$

Substituting f for σ^2

$$4f^3 - 6f\cdot\mu_2 + \mu_3 = 0 \qquad (6)$$

This cubic is easily solvable by Horner's method and the other constants may be determined from equations (4) and (5).

Geometric meaning of this transformation. One peculiarity of this type transformation is the infinite discontinuity at the origin. No matter what the modulus is or how far the center of gravity of the curve is removed from the origin, there is always, at least, one point of infinite discontinuity at the origin.

If we allow $\sigma=1$ and differentiate $\frac{1}{\sqrt{X}}e^{-\frac{(\sqrt{X}-a)^2}{2}}$ we shall be able to find the maxima, minima and points of inflection of the theoretical curve,

$$-\frac{1}{\sqrt{X}}e^{-\frac{(\sqrt{X}-a)^2}{2}}\cdot(\sqrt{X}-a)\cdot\frac{1}{2\sqrt{X}} + e^{-\frac{(\sqrt{X}-a)^2}{2}}\cdot\frac{1}{-2(\sqrt{X})^3} = 0, (\sqrt{X}-a) - \frac{1}{\sqrt{X}} = 0 ,$$

$$X = \frac{(a^2-2) \mp \sqrt{(a^2-2)^2-4}}{2}$$

We note that for $a<2$ there is no maximum, minimum or point of inflection. For $a=2$ there is a point of inflection at $X=1$, but their is neither maximum nor minimum. For $a>2$ we have one maximum and one minimum. For $a=4$, $X=13.92$ or 0.07.

This indicates that it is necessary to have the origin back a distance at least 4σ in the negative direction from the origin in order that all the end values of the normal curve may be included in the positive field. If the origin is not back far enough, the negative end, on being squared, will distort the distribution in a most surprising way, as will be seen presently.

Figure 1 is the normal curve. In case $a=\sigma$ there is not the slightest resemblance between the elementary normal curve and the transformed curve

(fig. 6). When $x=0$, $y=\infty$. From this point there is a continuous curve without maximum, minimum or point of inflection.

When $a=2\sigma$ the curve is hardly recognizable as related to the normal curve, as shown in figure 5, there is neither maximum nor minimum. At $x=1$, there is a point of inflection with the inflectional tangent parallel to the x-axis.

Figure 7 illustrates the distortion for $a=3\sigma$. In this case the maximum is at 6.89 and the minimum at 0.11. Even here the origin is not back far enough.

If $a\geq 4\sigma$ the curve presents the general appearance of a much flattened normal curve as shown in figure 8.

It is therefore necessary, if we are to expect the transformed curve to be similar in general appearance to the fundamental normal curve, that the centroid of the normal curve must be at a distance of at least 4σ above the origin, otherwise our transformed curve will be utterly different in general appearance from the normal curve.

It might well be expected that the weights of 5082 men who are five feet, nine inches tall and are between the ages of twenty and twenty-four years of age would be distributed according to some transformed curve. I have accordingly selected an illustration from the Medico-Actuarial Mortality Investigation, vol. I, page 41.

Sheppard's corrections were applied to the moments calculated about the centroid because of high contact at the ends of the range.

If μ_1, μ_2, μ_3, μ_4 be written for the adjusted moments, it was found that

$$\mu_1 = 0$$

$$\mu_2 = 190.08565$$

$$\mu_3 = 2203.6126$$

$$\mu_4 = 174,557.2162$$

The mean is at 150.096418. The calculated theoretical normal frequency distribution is given in Table I.

The class mark of each frequency group is given in column I. The difference between the centroid and the class mark X is given in column II. Column II, divided by the standard deviation σ, is given in column III. The fractional area under the curve between 0 and $\frac{X-M}{\sigma}$, as it is tabulated in books on Probability*, is in column IV. The entire area is given in column V, the calculated frequency in column VI, the observed frequency in column VII, and the residuals in column VIII.

TABLE I.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.
97.5	52.5964	3.81487	0.49993	2541	1	1	0
102.5	47.5964	3.45222	0.49972	2539.5	3	...	3
107.5	42.5964	3.08956	0.49928	2537	12	1	11
112.5	37.5964	2.72691	0.49683	2525	30	2	28
117.5	32.5964	2.36425	0.49091	2495	69	26	43
122.5	27.5964	2.0016	0.47728	2426	142	94	48
127.5	22.5964	1.63894	0.44384	2284	256	248	8
132.5	17.5964	1.27628	0.39906	2028	404	488	84
137.5	12.5964	0.91363	0.31954	1624	561	724	163
142.5	7.5964	0.55097	0.20916	1063	683	746	63
147.5	2.5964	0.18832	0.07468	380	731	817	86
152.5	2.4036	0.17434	0.06919	351			
157.5	7.4036	0.53691	0.20432	1039	678	600	78
162.5	12.4036	0.89964	0.31584	1605	566	513	53
167.5	17.4036	1.26230	0.39657	2015	410	315	95

* Davenport, Statistical Methods, p. 119.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.
172.5	22.4036	1.62495	0.44805	2277	262	208	54
177.5	27.4036	1.987613	0.47657	2422	155	108	47
182.5	32.4036	2.35027	0.49062	2493	71	75	4
187.5	37.4036	2.71292	0.49668	2524	31	38	7
192.5	42.4036	3.07558	0.49895	2536	12	28	16
197.5	47.4036	3.43828	0.49971	2539.5	4	18	14
202.5	52.4036	3.80089	0.49993	2540.6	1	17	16
212.5	57.4036	4.1635	0.5	2541	.4	5	4.6
217.5	62.4036	Beyond the tables.				4	4.
222.5						2	2.
227.5						1	1
232.5						2	2
237.5						..	0
242.5						..	0
247.5						1	1
						..	0

The normal function does not describe this frequency distribution at all , as is obvious from the above table.

Before calculating the theoretical distribution in case of squares it is necessary to determine the values of the constants a and σ. Substituting the values of μ_2 and μ_3 into equation (6)

$$4f^3 - 6\times190.08565f + 2203.6126 = 0.$$

It is easily shown that this equation has three real roots of which two are positive and one is negative. The negative value would make a^2 negative, the larger positive value would make a smaller than σ. A distribution of this sort is shown in fig. VI. The central value is, therefore, the appropriate one.

$$f = 1.9585.$$

Since $f = \sigma^2$ $\qquad\qquad \sigma = \pm 1.3987.$

Substituting the value of σ in equation (4)

$$a^2 = 23.74563,$$

$$a = \pm 4.8759.$$

With a no larger as compared to σ we could hardly expect an excellent fit.

In table II, an attempt was made to fit this observed distribution by means of a function where $X = x^2$, x's being the variates of the theoretical normal distribution. Column I contains the class marks; col. II, the distance from the origin to the median of the transformed curve plus the deviation from that median, a^2+X-M; col. III, $\frac{\sqrt{a^2+X-M}-a}{\sigma} = \frac{x}{\sigma}$; col. IV, the fractional area between 0 and $\frac{x}{\sigma}$*; col. V, fractional area times the number of frequencies; col. VI, the calculated frequencies; col. VII, the observed frequencies, and col. VIII the residuals.

TABLE II.

Above the median.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.
152.5	28.1367	0.30635	0.12032	611.4	717.	817.	100.
157.5	33.1367	0.62591	0.23431	1191.	580.	600.	20.
162.5	38.1367	0.92912	0.32357	1644.	453.	513.	60.
167.5	43.1367	1.20967	0.38365	1948.	304.	315.	11.
172.5	48.1367	1.4743	0.42979	2184.	236.	208.	28.
177.5	53.1367	1.7256	0.45778	2326.	142.	108.	34.
182.5	58.1367	1.96529	0.47530	2415.	89.	75.	14.
187.5	63.1367	2.19482	0.48591	2469.	54.	38.	16.
192.5	68.1367	2.4157	0.49214	2501.	32.	28.	4.
197.5	73.1367	2.62815	0.49569	2519.	18.	18.	0.
202.5	78.1367	2.8337	0.49769	2529.	10.	17.	7.

* Davenport, Statistical methods, pages 119-125.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.
207.5	83.1367	3.03274	0.49880	2535.	6.	5.	1.
212.5	88.1367	3.22592	0.49937	2537.	2.	4.	2.
217.5	93.1367	3.41355	0.49968	2539.	2.	2.	0.
222.5	98.1367	3.59648	0.49983	2540.	1.	1.	0.
227.5	103.1367	3.7749	0.49992	2540.5	0.5	2.	1.5
232.5	108.1367	3.94702	0.49996	2540.7	0.2	..	0.2
237.5	113.1367	4.11732	Beyond the tables.			..	0.
						1.	1.

Below the median.

147.5	23.1367	0.04704	0.01875	96.			
142.5	18.1367	0.44126	0.17048	866.	770.	746.	24.
137.5	13.1367	0.96625	0.33302	1692.	826.	724.	102.
132.5	8.1367	1.4467	0.425598	2163.	471.	488.	17.
127.5	3.1367	2.21984	0.48679	2474.	311.	248.	63.
122.5	-1.8633	imaginary.				94.	94.
						26.	26.
						2.	2.
						1.	1.
						0.	0.
						1.	1.

This transformation describes the upper end of the distribution fairly well, toward the center the transformed curve is too low and below the median there is no fit, the values becoming imaginary after the fourth. The appearance of the imaginary variates at the lower end of the distribution shows that the origin is not back far enough, in other words, that part of the variates of the elementary normal distribution fall in the negative region. After squaring these variates they become positive and change the form of the curve.

III. TRANSFORMATION $X = x^3$.

In case of similar volumes it seems quite reasonable that we might have a distribution $X = x^3$, where each variate of the normal distribution is cubed. In case the variates distribute themselves in this fashion, the constants may be determined by method of moments, in a manner similar to that shown in section II.

$$\int_{-\infty}^{+\infty} X^n f(X)\,dX = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} x^{3n} e^{-\frac{(x-a)^2}{2\sigma^2}}\,dx.$$

The limits of the observed function may be taken from $-\infty$ to $+\infty$ as the cubes of the negative numbers remain negative.

First moment. $$g = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} x^3 e^{-\frac{(x-a)^2}{2\sigma^2}}\,dx$$

Let $x = x' + a$ $$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} (x+a)^3 e^{-\frac{x^2}{2\sigma^2}}\,dx$$

$$= 3a\sigma^2 + a^3$$

Second moment.

$$g^2 + \mu_2 = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} (x+a)^6 e^{-\frac{x^2}{2\sigma^2}}\,dx$$

$$= a^6 + 15a^4\sigma^2 + 45a^2\sigma^4 + 15\sigma^6.$$

Third moment.

$$g^3 + 3g\mu_2 + \mu_3 = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} (x+a)^9 e^{-\frac{x^2}{2\sigma^2}}\,dx$$

$$= a^9 + 36a^7\sigma^2 + 378a^5\sigma^4 + 1260a^3\sigma^6 + 945a\sigma^8\ .$$

Eliminating g $$\mu_2 = 3a^4\sigma^2 + 9a^2\sigma^4 + 15\sigma^6.$$

$$\mu_3 = 945a\sigma^8 + 99a^3\sigma^6 + 351a^5\sigma^4 + 27a^7\sigma^2 - 3a\mu_2(3\sigma^2 + a^2).$$

Equations of this sort are of too high a degree and too complicated to be of use in making an application to observed distributions.

Since the cubes of negative numbers are negative it would, perhaps, at first thought, seem that the position of the origin would not effect the transformation. This, however, is not true. When $x=0$ the transformed

curve
$$y = \frac{1}{3\sigma\sqrt[3]{X^2}\sqrt{2\pi}} \cdot e^{-\frac{(\sqrt[3]{X})^2}{2\sigma^2}}$$

is similar in some respects to
$$y = \frac{1}{2\sigma\sqrt{2\pi X}} \cdot e^{-\frac{(\sqrt{X})^2}{2\sigma^2}}$$

Both of these curves begin at ∞ when $x=0$, then descend very rapidly for a time, touching the x-axis at infinity. The fundamental difference between the two cases is that the case of cubes has a pair of curves beginning at infinity for $x=0$ and extending in opposite directions and intersecting the x-axis at $-\infty$ and $+\infty$. Figure no. 2 is a graph of this curve. It will be noted that in the case of squares there is a single curve similar in shape to this curve in the first quadrant.

It is necessary that $a>\sigma\sqrt{8}$ in order that we may have a maximum or a minimum on either side of the y-axis. This function, as well as the function discussed in section II, has an infinite discontinuity at the origin. With regard to this infinite discontinuity, it does not seem likely that there would be any analogy in nature.

Let us determine the maxima and minima of
$$y = \frac{1}{3\sqrt[3]{X^2}\sqrt{2\pi}} \cdot e^{-\frac{(\sqrt[3]{X}-a)^2}{2}}$$
$$\frac{dy}{dX} = X^{-\frac{2}{3}} \cdot e^{-\frac{(\sqrt[3]{X}-a)^2}{2}} \left[-(\sqrt[3]{X}-a)X^{-\frac{2}{3}}\right] + e^{-\frac{(\sqrt[3]{X}-a)^2}{2}} \left(-\frac{2}{3}X^{-\frac{5}{3}}\right) = 0,$$
$$X^{\frac{2}{3}} - aX^{\frac{1}{3}} + 2 = 0,$$
$$X^{\frac{1}{3}} = \frac{a \mp \sqrt{a^2-8}}{2}.$$

For $a=\sigma$ or $a=2\sigma$, X is imaginary, there is, therefore, no maxima, minima or point of inflection. See figs. no. 3 and 4. For $a=3\sigma$, $X=1$ or 8 making a minimum at 1 and a maximum at 8. If the variates of the elementary normal curve are all positive, there will be a smooth flat curve, similar in form to the normal curve.

IV. TRANSFORMATION $X = a_1x + a_2x^2$.

This is a more general transformation than those treated in sections II and III. There is a certain propriety in using the first two or three terms of a Taylor's expansion. In this section, we will consider only the first two terms of the expansion. It is necessary that a_2 shall be small as compared to a_1 otherwise the curve will be very much distorted from a normal curve, since the theoretical distribution is here calculated from the median of the observed distribution.

It is equally general to make $\sigma = \frac{1}{\sqrt{2}}$ or $\sigma\sqrt{2}=1$, and to replace $\frac{x}{\sigma\sqrt{2}}$ by ξ, thus using Edgeworth's notation. Since by hypothesis a_2 is small compared to a_1, let $\frac{a_2}{a_1} = \kappa$ where κ is small. Then

$$X = a(\xi + \kappa\xi^2)$$

and
$$dX = a(1 + 2\kappa\xi)d\xi;$$

then
$$\int_{-\infty}^{+\infty} e^{-\xi^2} d\xi = \int e^{-\frac{-1\mp\sqrt{1+\frac{4\kappa X}{a}}}{4\kappa}} \cdot \frac{1}{a\left(1+2\kappa\left(\frac{-1\mp\sqrt{1\pm\frac{4\kappa X}{a}}}{2\kappa}\right)\right)} dX.$$

We shall denote the moments about the median as M_1, M_2, M_3, M_4 respectively, and μ_1, μ_2, μ_3, μ_4 denote the moments about the center of gravity.

First moment. $M_1 = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} a(\xi + \kappa\xi^2)e^{-\xi^2} d\xi$

$$= \frac{1}{2}a\kappa.$$

Second moment. $M_2 = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} a^2(\xi + \kappa\xi^2)^2 e^{-\xi^2} d\xi$

$$= \frac{1}{2}a^2 + \frac{3}{4}a^2\kappa^2.$$

Third moment. $M_3 = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} a^3(\xi + \kappa\xi^2)^3 e^{-\xi^2} d\xi$

$$= \frac{9}{4}\kappa a^3 + \frac{15}{8}\kappa^3 a^3.$$

$$\mu_2 = M_2 - M_1^2 = \frac{1}{2}a^2 + \frac{3}{4}a^2\kappa^2 - \frac{1}{4}a^2\kappa^2$$

$$= \frac{1}{2}a^2(1 + \kappa^2).$$

$$\mu_3 = M_3 - 3M_1\mu_2 - M_1^3 = a^3\kappa\left(\frac{3}{2} + \kappa^2\right).$$

$$\beta_1^* = \frac{\mu_3^2}{\mu_2^3} = \frac{[\alpha^3\kappa(\frac{3}{2} + \kappa^2)]^2}{[\frac{1}{2}\alpha^2(1 + \kappa^2)]^3} \qquad (1)$$

$$= \frac{\alpha^6\kappa^2(\frac{9}{4} + 3\kappa^2 + \kappa^4)}{\alpha^6\frac{1}{8}(1 + 3\kappa^2 + 3\kappa^4 + \kappa^6)}.$$

From (1) $\quad \kappa^2(\frac{9}{4} + 3\kappa^2 + \kappa^4) - \beta_1\frac{1}{8}(1 + 3\kappa^2 + 3\kappa^4 + \kappa^6) = 0.$

Let $\chi = \kappa^2$. Then

$$8\chi(\frac{9}{4} + 3\chi + \chi^2) - \beta_1(1 + 3\chi + 3\chi^2 + \chi^3) = 0,$$

$$18\chi + 24\chi^2 + 8\chi^3 - \beta_1(1 + 3\chi + 3\chi^2 + \chi^3) = 0. \qquad (2)$$

$$X = \alpha(\xi + \kappa\xi^2),$$

$$\xi = \frac{-1 \pm \sqrt{1 + \frac{4\kappa X}{\alpha}}}{2\kappa} \qquad (3).$$

Since $\xi = \frac{X}{\sigma\sqrt{2}}$ it is highly improbable for $|\xi|$ to be greater than 5. In fact we shall prove presently, that negative values of ξ lie between 0 and -5. This transformation will be applied to the data used in section II. To do this it is necessary to get first the values of α and κ.

From (1) $\quad \beta = \frac{\mu_3^2}{\mu_2^3} = \frac{4855908.4908}{6868280.078} = 0.707005.$

$$7.292995\chi^3 + 21.878985\chi^2 + 15.878985\chi - 0.707005 = 0,$$

$$\chi^3 + 3\chi^2 + 2.177292\chi - 0.096943 = 0,$$

$$\chi = 0.0418.$$

$$\kappa^2 = \chi,$$

$$\kappa = \pm\ 0.2044504.$$

$$\mu_2 = \frac{1}{2}\alpha^2(1 + \kappa^2),$$

$$\alpha^2 = 364.9177385,$$

$$\alpha = \pm\ 19.1028.$$

In the following table III, X as given in column I is calculated from the median, ξ of column II is calculated by substituting the values of X, α, and κ into equation (3), $\varphi(\xi)$ of column III is taken from Czu-

* Phil. Trans. Roy. Soc. 1895, vol. 186, p. 351.

ber*. $\varphi(\xi)\times\frac{5082}{2}$ is found in column IV. The calculated theoretical frequency is given in column V, the observed frequency, in column VI, and the residuals, in column VII.

TABLE III.

Above the median.

I.	II.	III.	IV.	V.	VI.	VII.
3.7087	0.187014	0.208601	530	729	817	90
8.7087	0.419821	0.44732	1136	606	600	6
13.7087	0.635216	0.63128	1604	468	513	45
18.7087	0.836317	0.76308	1939	335	315	20
23.7087	1.022475	0.851788	2164	225	208	17
28.7087	1.20572	0.91181	2317	153	108	45
33.7087	1.376364	0.94846	2410	93	75	18
38.7087	1.540841	0.97068	2466	56	38	18
43.7087	1.698141	0.98368	2499	33	28	5
48.7087	1.849278	0.99106	2518	19	18	1
53.7087	1.996378	0.99522	2529	11	17	6
58.7087	2.138178	0.99748	2534	5	5	0
63.7087	2.275739	0.998705	2538	4	4	0
68.7087	2.409378	0.999343	2539	1+	2	1
73.7087	2.539985	0.999670	2540	1	1	0
78.7087	2.66654	0.999837	2541.-	1-	2	1
83.7087	2.78974	0.9999199	2541.-	0	0	0
88.7087				0	0	0
93.7087				0	1	1
		Below the median.				
1.2913	0.06857	0.077627	197			
6.2913	0.355268	0.384668	977	780	746	34
11.2913	0.687809	0.66926	1700	723	724	1

* Wahrscheinlichkeitsrechnung, page 385.

I.	II.	III.	IV.	V.	VI.	VII.
16.2913	1.10381	0.88029	2237	537	488	49
21.2913	1.718048	0.984883	2503	266	248	16
26.2913	imaginary				94	94
31.2913	"				26	26
36.2913	"				2	2
41.2913	"				1	1
46.2913	"				..	..
51.2913	"				1	1

This function fits the observed distribution better than the normal frequency function or the transformation discussed in section II for the portion of the range which gives real values. For the positive end of the range and the first four frequency groups of the negative range, this function describes the observed distribution fairly well. From the fourth term on,the terms on the negative side are imaginary. Professor Edgeworth states* "that the difficulty from occurence of imaginary values is not apprehended". In this example imaginary values of a and κ do not occur, but there is difficulty from the values of ξ becoming imaginary.

Here another problem represents itself, namely within what interval must ξ remain in order that all negative values of ξ will remain in the negative region for the transformed function X.

$$X = 19.1028\xi + 3.8275\xi^2$$

but $\xi = -t$ where t is positive. When is $-19.1028t + 3.8275t^2 < 0$?

$$3.8275t^2 < 19.1028t\ ,$$

$$t < 5.$$

Therefore ξ must lie within the interval $0 < \xi < -5$ so that any negative value of ξ will give a negative X.

* International Congress of Mathematicians,1912,vol.II, p.431.

Pearson's criterion of best fit* applied for real values of ξ gives $\chi'^2=44.5$ and the probability $P=0.0053$ that deviations as great as or greater than these would occur under random sampling, where

$$\chi^2 = \frac{(\text{squares of differences of theoretical and observed frequencies})}{(\text{theoretical frequencies})}$$

Another transformation which might be of interest for some special cases is to replace the variable x by X where $X=x+x^2$, $(1+2x)dx=dX$, $dx=\frac{dX}{1+(-1\pm\sqrt{1+4X})}$.

$$\int e^{-\frac{(x-a)^2}{2\sigma^2}}dx = \int \frac{1}{-1\pm\sqrt{1+4X}}\cdot e^{-\frac{[(-1\pm\sqrt{1+4X})-a]^2}{2\sigma'^2}}dX$$

It is necessary that the median of the normal curve should be sufficiently far above the origin so that all variates be positive.

Replacing each x by X and solving for the constants by method of moments we get

$$\frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty X^n e^{-\frac{[(1\pm\sqrt{1+4X})-a]^2}{2\sigma'^2}}\cdot\frac{1}{1\pm\sqrt{1+4X}}\cdot dX = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty}(x+x^2)^n e^{-\frac{(x-a)^2}{2\sigma^2}}dx.$$

First moment.

$$\begin{aligned}
g &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty}(x+x^2)e^{-\frac{(x-a)^2}{2\sigma'^2}}dx\ ,\\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty}[(x'+a)+(x'+a)^2]e^{-\frac{x^2}{2\sigma'^2}}dx,\\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty}[(x+a+\tfrac{1}{2})^2-\tfrac{1}{4}]e^{-\frac{x^2}{2\sigma^2}}dx,\\
g &= (a+a^2)+\sigma^2. \qquad \text{I.}
\end{aligned}$$

Second moment.

$$\begin{aligned}
g^2+\mu_2 &= \frac{1}{\sigma\sqrt{2\pi}}\int[(x+a+\tfrac{1}{2})^2-\tfrac{1}{4}]^2e^{-\frac{x^2}{2\sigma^2}}dx\\
&= 3\sigma^4+\sigma^2(6a^2+6a+1)+a^4+2a^3+a^2 \qquad \text{II.}
\end{aligned}$$

Third moment.

$$\begin{aligned}
g^3+3g\mu_2+\mu_3 &= \int_{-\infty}^{+\infty}[(x+a+\tfrac{1}{2})^2-\tfrac{1}{4}]^3e^{-\frac{x^2}{2\sigma^2}}dx\\
&= 15\sigma^6+3\sigma^4(15a^2+15a+3)+\sigma^2(15a^4+30a^3+18a^2+3a)+a^6+a^5+3a^4+a^3. \qquad \text{III.}
\end{aligned}$$

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* Biometrika, vol. I, pp. 155-161.

Substituting the value of g from equation I into equation (2) and (3)

$$\mu_2 = 2\sigma^4 + \sigma^2(4a^2 + 4a + 1) \qquad (4)$$

$$\mu_3 = 11\sigma^6 + (9a^2 + 9a + 2)\cdot 3\sigma^4 \qquad (5)$$

$$= 11\sigma^6 + 2(4a^2 + 4a + 1)\cdot 3\sigma^4 + (a^2 + a)\cdot 3\sigma^4.$$

From equation (4)

$$4a^2 + 4a + 1 = \frac{\mu_2 - 2\sigma^4}{\sigma^2} \qquad (6)$$

Substituting the value of the left hand member of (6) in (5)

$$\mu_3 = 11\sigma^6 + 6\sigma^4\left[\frac{\mu_2 - 2\sigma^4}{\sigma^2}\right] + \left[\frac{\mu_2 - 2\sigma^4}{4\sigma^2}\right]\cdot 3\sigma^4 - \frac{3}{4}\sigma^4,$$

$$4\mu_3 + 10\sigma^6 + 3\sigma^4 - 27\sigma^2\mu_2 = 0$$

Putting $\sigma^2 = f$,

$$10f^3 + 3f^2 - 27f\mu_2 + 4\mu_3 = 0,$$

$$f^3 + 0.3f^2 - 2.7f\mu_2 + .4\mu_3 = 0.$$

Since the equation in a is one of the second degree the solution of the two equations is facilitated by substituting the value of $4a^2+4a+1$ from equation (6) into equation (5).

V. TRANSFORMATION $X = a_1\xi + a_2\xi^2 + a_3\xi^3$.

This type of transformation is more general, perhaps, and will change the general form of the curve. While ξ^2 tends to pile the variates near the origin and in the negative region ξ^3 has a more flattening and distributing effect.

If $y=f(X)$, then where $f(\xi)$ is the normal function with the standard deviation $\sigma=\frac{1}{\sqrt{2}}$ or $\sigma\sqrt{2}=1$ and $X=a(\xi+\kappa\xi^2+\lambda\xi^3)$, then the n^{th} moment about the median is

$$\int_{-\infty}^{\infty}X^n f(X)dX = \int_{-\infty}^{\infty}a^n(\xi+\kappa\xi^2+\lambda\xi^3)^n e^{-\xi^2}d\xi.$$

The required constants a, κ and λ can be solved for by method of moments.

The moments are calculated from the median. For convenience we shall use Edgeworth's notation*. M_1, M_2, etc., represent the moments about the median.

$$M_1 = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} a(\xi + \kappa\xi^2 + \lambda\xi^3)e^{-\xi^2}d\xi$$
$$= \frac{1}{2}a\kappa.$$
$$M_2 = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} a^2(\xi + \kappa\xi^2 + \lambda\xi^3)^2 e^{-\xi^2}d\xi$$
$$= a^2\left(\frac{1}{2} + \frac{3}{4}\kappa^2 + \frac{3}{2}\lambda + \frac{15}{8}\lambda^2\right).$$
$$M_3 = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} a^3(\xi + \kappa\xi^2 + \lambda\xi^3)^3 e^{-\xi^2}d\xi$$
$$= a^3\left(\frac{9}{4}\kappa^2 + \frac{15}{8}\kappa^3 + \frac{45}{4}\kappa\lambda + \frac{315}{16}\kappa\lambda^2\right)$$

$$\mu_1 = M_1 = \frac{1}{2}a\kappa. \tag{1}$$

$$\mu_2 = M_2 - M_1^2 = \frac{a^2}{2}\left(1 + \kappa^2 + 3\lambda + \frac{15}{4}\lambda^2\right) \tag{2}$$

$$\mu_3 = M_3 - 3M_1\mu_2 - M_1^3 = a^3\kappa\left(\frac{3}{2} + \kappa^2 + 9\lambda + \frac{135}{8}\lambda^2\right) \tag{3}$$

Since $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$ and $\eta^{**} = \frac{\mu_4}{\mu_2^2} - 3$ (4)

we may solve for the constants a, κ, λ.

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{8\kappa^2(\frac{3}{2} + \kappa^2 + 9\lambda + 135\lambda^2)^2}{(1 + \kappa^2 + 3\lambda + \frac{15}{4}\lambda^2)^3}.$$

$$\eta = \frac{\mu_4}{\mu_2^2} - 3 = \frac{4(6\kappa^2+3\lambda+3\kappa^4+54\kappa^2\lambda+27\lambda^2+135\kappa^2\lambda^2+\frac{405}{4}\lambda^3+\frac{1215}{8}\lambda^4}{(1 + \kappa^2 + 3\lambda + \frac{15}{4}\lambda^2)^2}$$

We may use $\chi = \kappa^2$,

$$8\chi\left(\frac{3}{2} + \chi + 9\lambda + \frac{135}{8}\lambda^2\right)^2 - \beta_1\left(1 + \chi + 3\lambda + \frac{15}{4}\lambda^2\right)^3 = 0 \tag{5}$$

$$4\left(6\chi+3\lambda+3\chi^2+54\chi\lambda+27\lambda^2+135\chi\lambda^2+\frac{405}{4}\lambda^3+\frac{1215}{4}\lambda^4\right) - \eta\left(1+\chi+3\lambda+\frac{15}{4}\lambda^2\right)^2 = 0 \tag{6}$$

These equations are not always so difficult to solve as they seem. This is due to the fact that κ and λ are usually small fractions. The higher powers will, therefore, become small. In trying out the numerical data used in sections II and IV, I made a first approximation, using only the first

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* Fifth International Congress of Mathematicians, Proceedings, vol. II, 1912, p. 430.

** The $\eta = \beta_2 - 3$ in Pearson's notation.

powers of κ and λ. Then from equations (5) and (6) we get:

$$18\chi - 0.707005(1 + 3\chi + 9\lambda) = 0 \qquad (7)$$

and

$$24\chi + 12\lambda - 1.833691(1 + 2\chi + 6\lambda) = 0 \qquad (8).$$

Solving for χ and λ

$$\lambda = 0.1015,$$

$$\chi = 0.0852.$$

Here λ and χ seem too large, so a second approximation was made. This was done by substituting for each λ and χ in equations (5) and (6) the value of λ and χ plus some new λ and χ say λ' and χ'. Thus $\lambda = 0.1015 + \lambda'$, and $\chi = 0.0852 + \chi'$. Only the first powers of χ' and λ' were retained because of the difficulty of solution of higher powers.

$$\chi' = -\ 0.030603\ ,$$

$$\lambda' = -\ 0.037525\ .$$

Then

$$\chi = \chi' + 0.08529$$

$$\chi = 0.054069$$

$$\lambda = \lambda' + 0.101529$$

$$\lambda = 0.07303.$$

Since $\chi = \kappa^2$,

$$\kappa = \pm\ 0.233679.$$

These values of κ and λ are substituted in equation (2) and a is solved for.

$$a^2 = 293.3694$$

$$a = \pm\ 17.1425.$$

These constants have been determined in a more precise way than in section IV. Because of the immense amount of labor that would be involved in solving a cubic equation by Horner's method for each value of ξ - and in choosing the root appropriate for the transformation and because κ, λ ,..

are a decreasing sequence, that is λ is smaller than κ, it was thought best to retain only the first and second powers of ξ.

From
$$X = a(\xi + \kappa\xi^2)$$
$$\xi = \frac{-1 \pm\sqrt{1 + \frac{4\kappa X}{a}}}{2\kappa} .$$

In table IV the theoretical frequency of this sort of transformation is given. Column I gives the value of X as calculated from the median; col. II gives ξ calculated from a, κ, X where $\xi = \frac{-1\pm\sqrt{1+\frac{4\kappa X}{a}}}{2\kappa}$; col. III, $\varphi(\xi)$; col. IV, $\varphi(\xi)\times\frac{5082}{2}$; col. V the theoretical frequency; col. VI the observed frequency; col. VII the residuals.

TABLE IV.

Above the median.

I.	II.	III.	IV.	V.	VI.	VII.
3.7087	0.20637	0.22956	583	831	817	14
8.7087	0.45896	0.48366	1229	646	600	46
13.7087	0.68844	0.66970	1702	473	513	40
18.7087	0.902113	0.79745	2026	324	315	9
23.7087	1.09965	0.88009	2236	210	208	2
28.7087	1.28715	0.93133	2367	131	108	23
33.7087	1.46496	0.961696	2444	77	75	2
38.7087	1.63496	0.979232	2488	44	38	6
43.7087	1.79579	0.989079	2513	25	28	3
48.7087	1.95321	0.9942523	2526	13	18	5
53.7087	2.101108	0.9970339	2533	7	17	10
58.7087	2.24568	0.998606	2537	4	5	1
63.7087	2.38564	0.9992582	2539	2	4	2
68.7087	2.52658	0.9996473	2540	1	2	1
73.7087	2.65404	0.999815	2541	1	1	0
78.7087	2.907407	0.99983	2541	0	2	2
83.7087	3.02934	0.9999809	2541	0	0	0

I.	II.	III.	IV.	V.	VI.	VII.
88.7087	3.14875	0.9999915	2541	0	0	0
93.7087	3.26559	0.999996	2541	0	1	0
		Below the median.				
1.2913	0.076707	0.097577	248			
6.2913	0.401854	0.43016	1093	845	746	99
11.2913	0.81323	0.74992	1905	812	724	88
16.2913	1.424588	0.956059	2429	524	488	36
21.2913	imaginary				248	248
26.2913	"				94	94
31.2913	"				26	26
36.2913	"				2	2
41.2913	"				1	1
46.2913	"				0	0
51.2913	"				1	1

The upper half of this transformed curve describes the given distribution very well. There is, however, absolutely no fit below the centroid, the values becoming imaginary after the third frequency group below the origin. It is to be observed that the negative values appear in this transformation sooner than in the other transformations. This is due to the fact that a is not large enough as compared with with κ.

Applying Pearson's criterion of fit to the part above the median, we find that

$$\chi'^2 = 22.47$$

and the probability that deviations as great as these would occur under random sampling is

$$P = 0.055 .$$

In conclusion, it may be fitting to say that an indefinite number of

transformations are possible. The difficulty is in selecting, a priori, the appropriate transformation for the particular data given. The transformed functions considered in this paper have advantages in being capable of application to data which in general appearance deviates from the normal curve. This fact is shown by the graphs of the transformed function (figs. 1-8).

The normal function does not describe the data considered in this paper. While the transformed function discussed in section II is better for part of the data, there are other parts which this function does not describe.

The Pearsonian criterion of fit could not be applied as the tables do not extend far enough, in other words there is little probability that such great deviation is due to random sampling. If the part of the theoretical distribution above the median is considered by itself the probability that such deviation is due to random sampling is $P = 0.000001$. Thus even the part above the median is a very bad fit.

The function considered in section IV gives a much better description except where the imaginary numbers appear.

The theoretical distribution of section V describes the part above the median very well, as was found by application of Pearson's criterion of fit. Below the median there are only three real values, the remaining values being imaginary.

The disappearance of the difficulty of imaginary values here obtained is to be expected when x is smaller compared to a than it is in this problem. This condition is doubtless realized for cases which deviate but slightly from the normal. It seems difficult to determine, a priori, other

conditions under which $\kappa \xi^2$ is sufficiently small so that we entirely avoid the occurence of imaginary values of ξ to correspond to numerically large negative values of X.

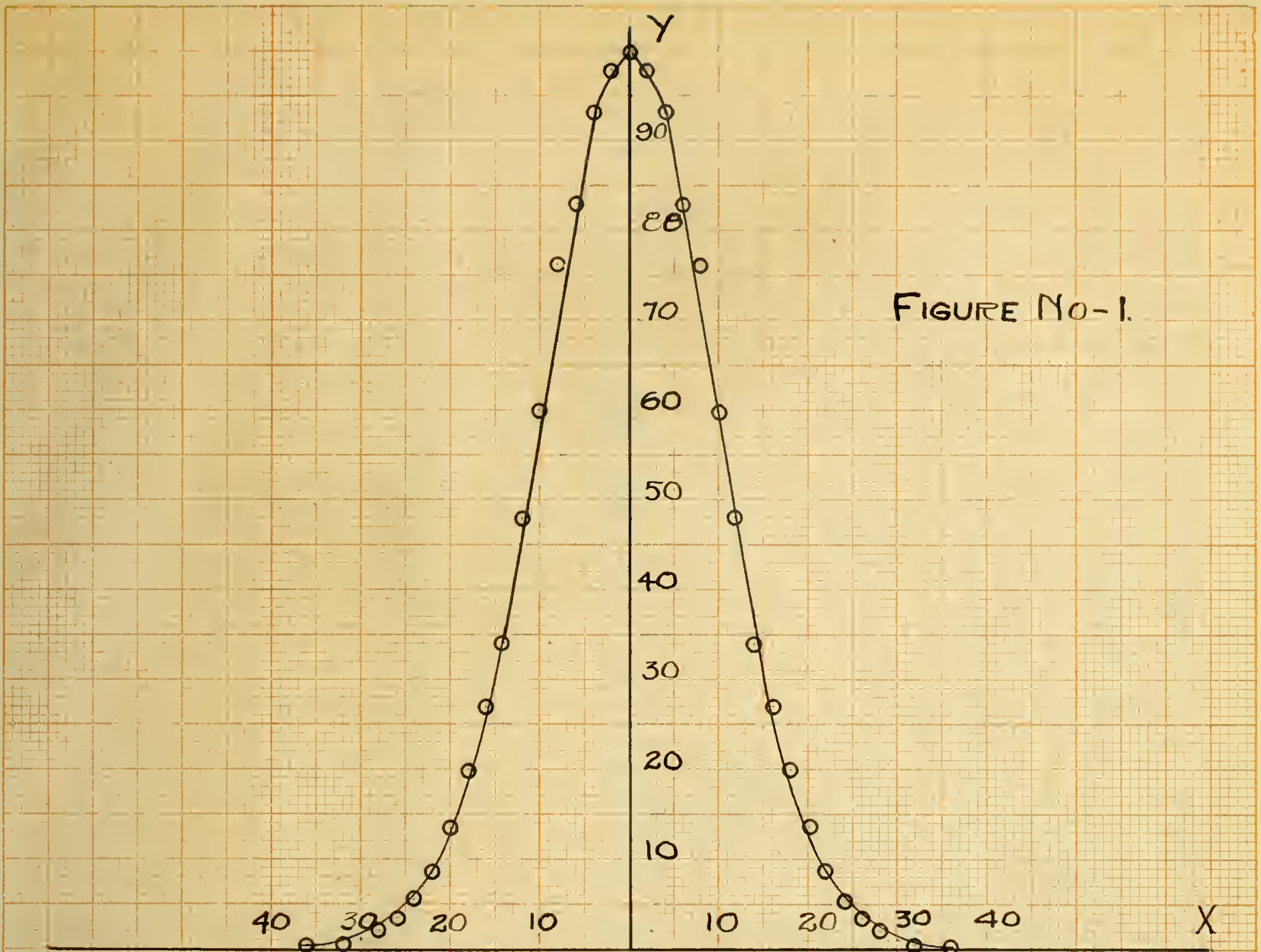

Figure No-1.

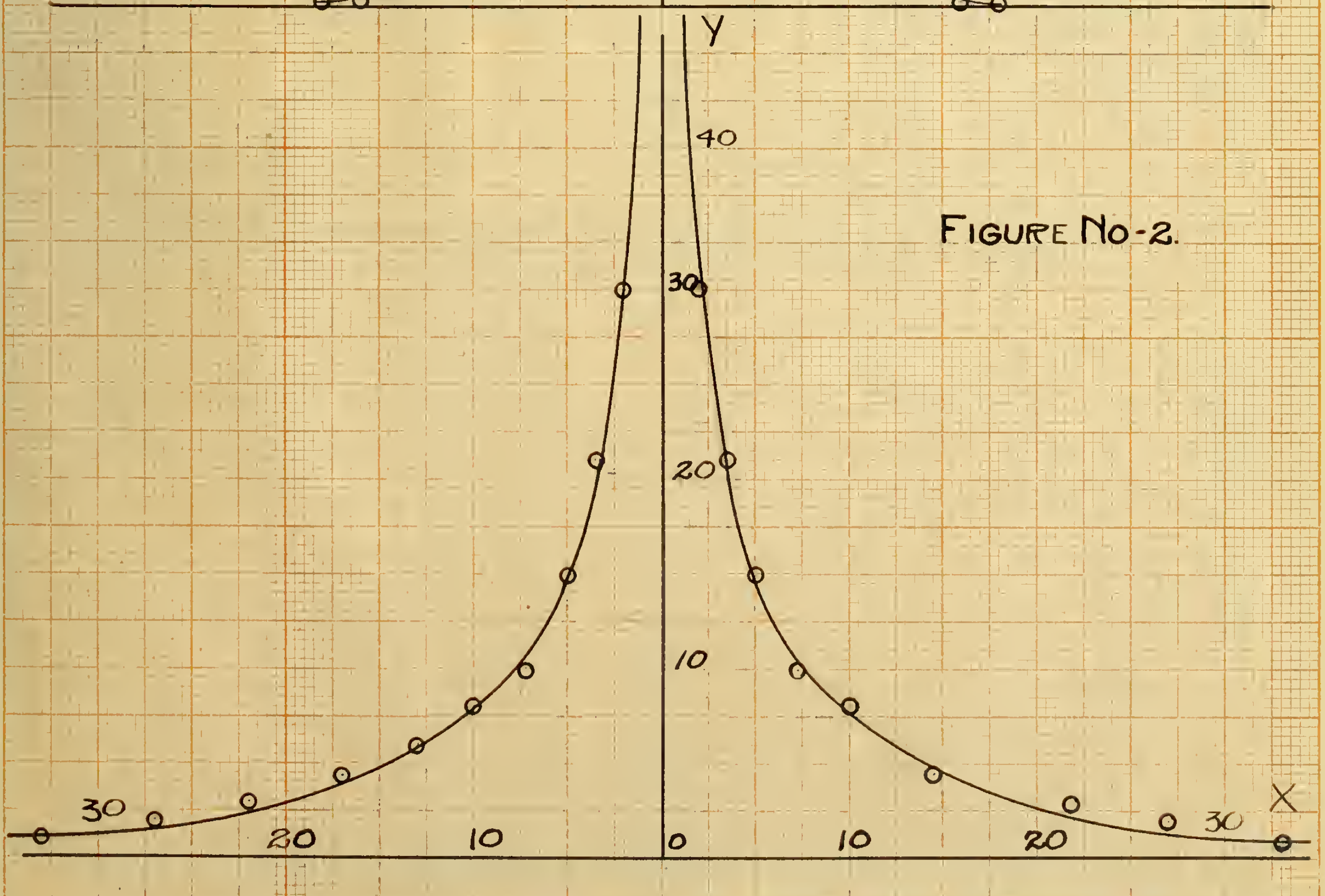

Figure No-2.

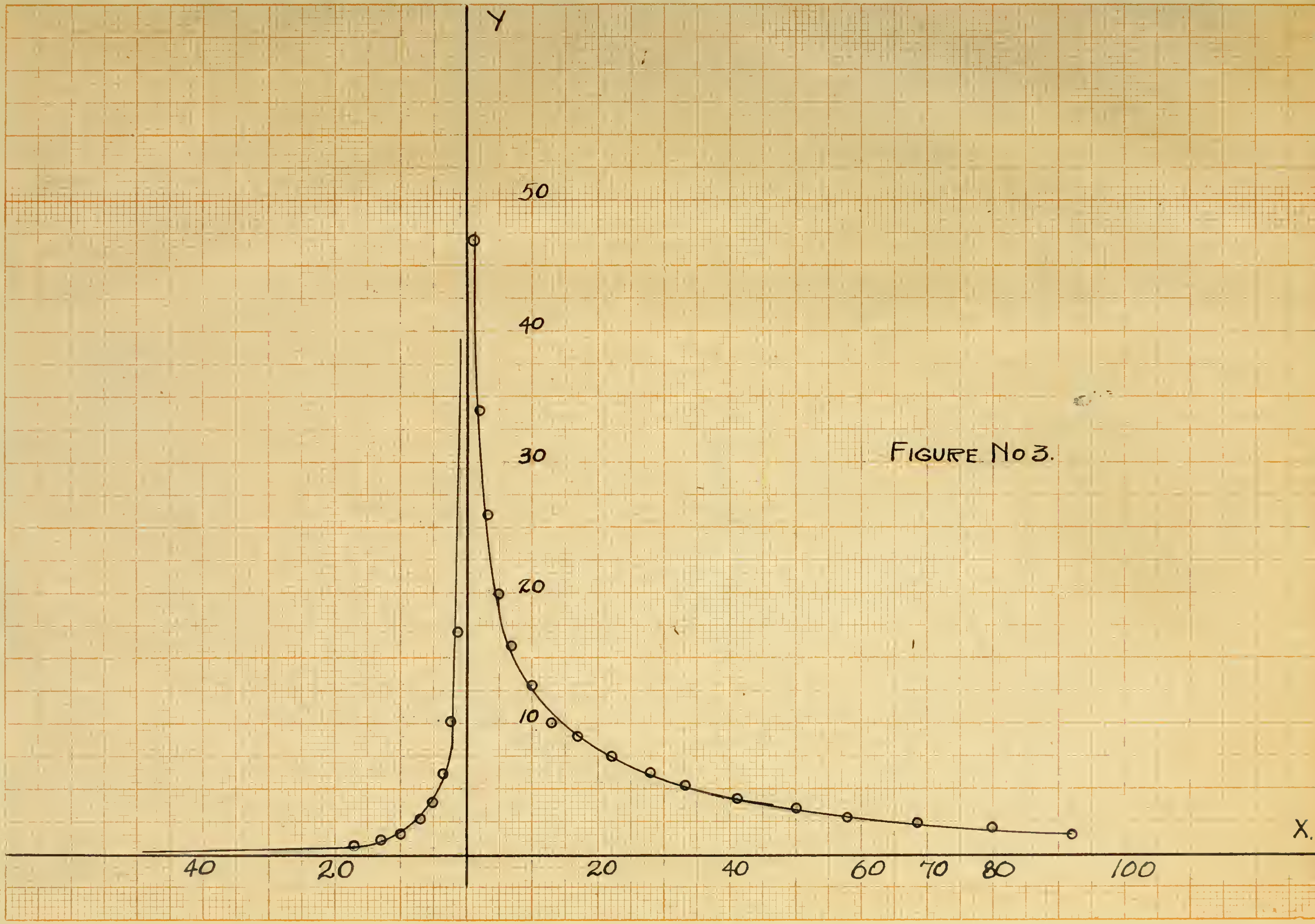

Figure No 3.

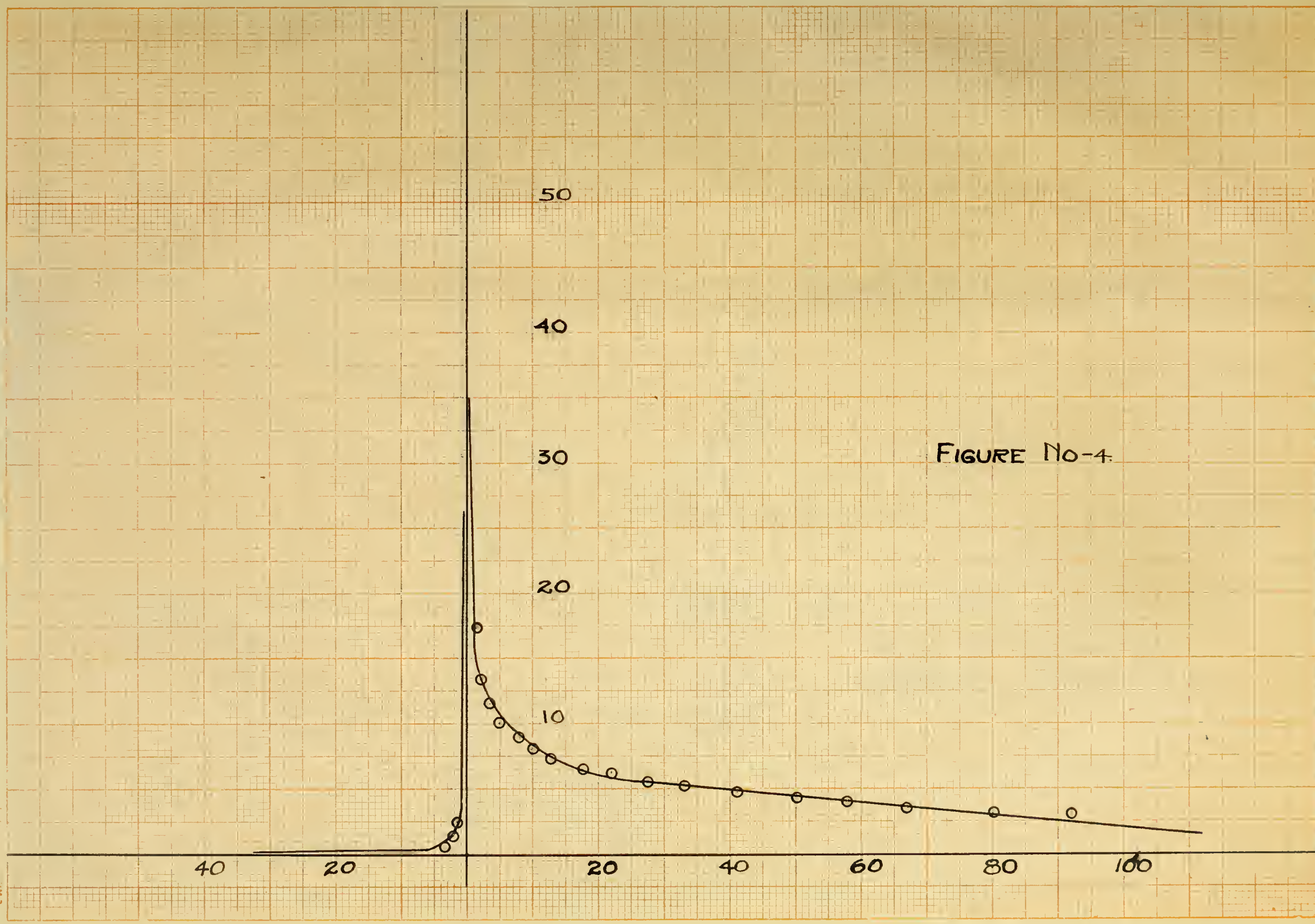

FIGURE No-4
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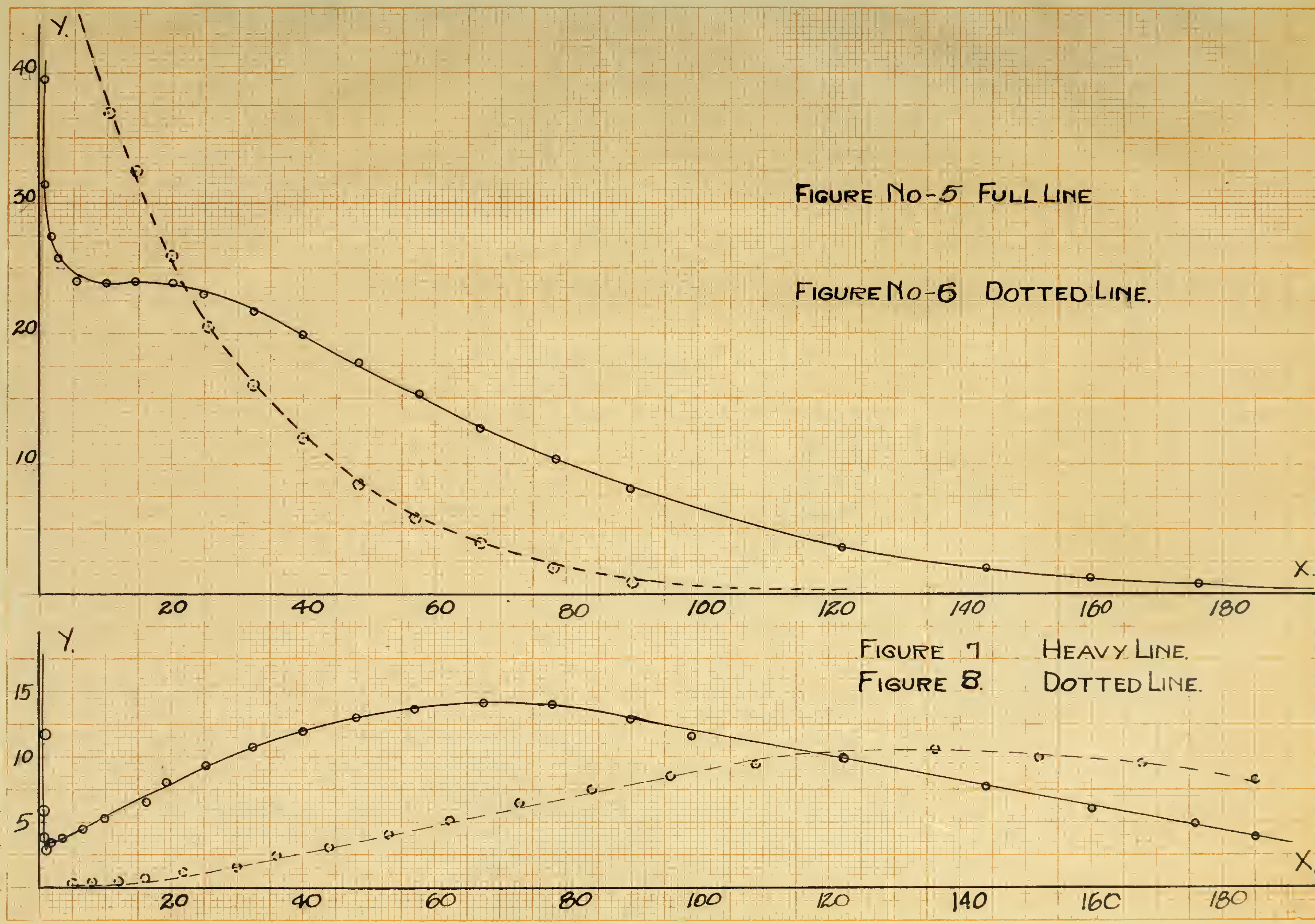

Figure No-5 Full Line
Figure No-6 Dotted Line.
Figure 7 Heavy Line.
Figure 8. Dotted Line.
Y.
X.
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160
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